NON-ARCHIMEDEAN TRANSPORTATION PROBLEMS AND KANTOROVICH ULTRA-NORMS

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ABSTRACT. We study a non-archimedean (NA) version of transportation problems and introduce naturally arising ultra-norms which we call *Kantorovich ultra-norms*. For every ultra-metric space and every NA valued field (e.g., the field $\mathbb{Q}_p$ of p-adic numbers) the naturally defined inf-max cost formula achieves its infimum. We also present NA versions of the Arens-Eells construction and of the integer value property. We introduce and study *free NA locally convex spaces*. In particular, we provide conditions under which these spaces are normable by Kantorovich ultra-norms and also conditions which yield NA versions of Tkachenko-Uspenskij theorem about free abelian topological groups.

CONTENTS

1. INTRODUCTION

Kantorovich norm [7] plays a major role in various areas of mathematics, economics and computer science (see [4, 6, 12, 13, 14, 20, 23, 24]), for instance, in Monge-Kantorovich transportation problem. The seminorms that determine the topology of the free real locally convex space are in fact Kantorovich seminorms (see [6, 12, 13, 20]). Uspenskij [22] provided a simplified formula for these seminorms.

In this paper we deal with discrete transportation problems. In Subsection 2.2 we present a slightly more flexible ("democratic") approach to the classical Kantorovich problem. This approach is related to the *transshipment problem.* Continuing in this direction, in Section 3 we study *non-archimedean transportation problems*. Since the term *non-archimedean* appears many times in this work, we write shortly: NA.

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In Section 4 we present an NA version of the Arens-Eells embedding (Theorem 4.3). We introduce the naturally arising Kantorovich ultra-(semi)norms, defined for an ultra-(pseudo)metric space (X,d) on free vector spaces $L_{\mathbb{F}}(X)$ via min-max formula. Theorem 4.3 shows that for an arbitrary NA valued field $\mathbb{F}$ and for any $u \in L_{\mathbb{F}}(X)$ the value of the Kantorovich ultra-(semi)norm $||u||$ can be approximated as

$$||u|| = \inf \left\{ \max_{1\leq i\leq k} |s_i| d(x_i,y_i) : u = \sum_{i=1}^{k} s_i(x_i - y_i),\ x_i, y_i \in \operatorname{supp}(u),\ s_i \in \mathbb{F} \right\},$$

where $\operatorname{supp}(u)$ is the support of u (see Notation 2.1).

Note that the analogous property in the archimedean case does not hold in general. Indeed, it is no longer true when $\mathbb{F} = \mathbb{C}$, the field of complex numbers, in contrast to the case $\mathbb{F} = \mathbb{R}$ (see [5, 25] and Remark 2.3.3 below).

The infimum in Theorem 4.3 is, in fact, a minimum. This refinement, which comes from Min-attaining Theorem 5.6, provides another contrast to the archimedean case. Indeed, in the Appendix we give an example in which the infimum is not attained for $\mathbb{F} = \mathbb{Q}(i)$. Another refinement concerns the coefficients (the *G-value property*), that is, it is enough to take the coefficients from the additive subgroup G_u of $\mathbb{F}$ generated by the normal coefficients λ_i of $u = \sum_{i=1}^{n} \lambda_i x_i$. Namely, we show that

$$||u|| = \min \left\{ \max_{1\leq i,j\leq m} |c_{ij}| d(x_i,x_j) : c_{ij} \in \mathbb{F},\ \forall i : 1 \leq i \leq n\ \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\},$$

such that all coefficients c_{ij} belong to G_u. Note that a matrix $(c_{ij}) \in \mathbb{F}^{m\times m}$ satisfies the equations $\sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i\ \forall i : 1 \leq i \leq n$ if and only if $u = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}(x_i - x_j)$. As a particular case we get an NA generalization of the so-called *integer value property* (well known in case $\mathbb{F} = \mathbb{R}$).

Probably one can encounter a variety of min-max optimization problems when dealing with the Kantorovich ultra-norms. It is worth noting that different algorithms for solving such problems are known (see [3] for example).

In Section 6 we introduce the *free NA locally convex spaces* for NA uniform spaces. We describe their topologies in terms of Kantorovich ultra-seminorms (Theorem 6.2). We show that for an ultra-metric space (X,d) and a trivially valued field $\mathbb{F}$, the free NA locally convex space $L_{\mathbb{F}}(X,\mathcal{U}(d))$ (of the uniformity $\mathcal{U}(d)$ of d) is normable by the Kantorovich ultra-norm induced by d (Theorem 6.5). By Tkachenko-Uspenskij theorem (in the archimedean case $\mathbb{F} = \mathbb{R}$) the free abelian topological group $A(X)$ is a topological subgroup of $L(X)$. Using Ostrowski's classical theorem we prove that in case $\mathbb{F}$ is an NA valued field of zero characteristic, the uniform free NA abelian topological group $A_{\mathcal{NA}}(X,\mathcal{U})$ is a topological subgroup of $L_{\mathbb{F}}(X,\mathcal{U})$ if and only if the restricted valuation on $\mathbb{Q}$ is trivial (Theorem 6.13). For example, this is the case for the Levi-Civita field (Example 6.14).

2. Kantorovich norm

For a nonempty set X and a field $\mathbb{F}$ denote by $L_{\mathbb{F}}(X)$ the free $\mathbb{F}$-vector space on the set X. We simply write $L(X)$ in case $\mathbb{F} = \mathbb{R}$. Define $\overline{X} := X \cup \{\mathbf{0}\}$ where $\mathbf{0} \notin X$ is the zero element of $L_{\mathbb{F}}(X)$. The zero element of the field $\mathbb{F}$ is denoted by $0_{\mathbb{F}}$. Denote by $L^0_{\mathbb{F}}(X)$ the

kernel of the linear functional

$$L_{\mathbb{F}}(X) \to \mathbb{F},\ \sum_{i=1}^{n} \lambda_i x_i \mapsto \sum_{i=1}^{n} \lambda_i.$$

Notation 2.1. Every non-zero vector $u \in L_{\mathbb{F}}(X)$ has a *normal form* as follows: $u = \sum_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$, where $x_i \in X,\ \lambda_i \in \mathbb{F} \setminus \{0_{\mathbb{F}}\}\ \forall i : 1 \leq i \leq n$ and $x_i \neq x_j$ whenever $i \neq j$. If $u \in L^0_{\mathbb{F}}(X)$ then define the *support of* u as $\mathrm{supp}(u) := \{x_1, \dots, x_n\}$. Otherwise, let $\mathrm{supp}(u) := \{x_1, \dots, x_n, x_{n+1}\}$ where $x_{n+1} = \mathbf{0}$. We denote by $m := |\,\mathrm{supp}(u)|$ the length of the support, so m is either n or $n+1$. The support of $\mathbf{0}$ is $\{\mathbf{0}\}$. In what follows, by writing $u = \sum_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$ we mean that it is a normal form.

2.1. **Classical transportation problem.** Recall the following transportation problem from the historical work of Kantorovich [7]. Let (X, d) be a metric space and denote by $\mathbb{R}_{\geq 0}$ the set of non-negative reals. Suppose that a network of railways connects a number of production locations $x_1, \dots, x_n \in X$ with daily output of $\lambda_1, \dots, \lambda_n$ carriages of certain goods, respectively, to a number of consumption locations $y_1, \dots, y_m \in X$ with daily demand of $\mu_1, \dots, \mu_m$ carriages. So, we have $\sum_{i=1}^{n} \lambda_i = \sum_{j=1}^{m} \mu_j$, where λ_i, μ_j are positive. Let c_{ij} denote the real number transferred from point x_i to point y_j. We view the metric d as a cost function, and we want to minimize our total sum-cost. The value we are seeking is

$$\inf \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} d(x_i, y_j) : c_{ij} \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} c_{ij} = \mu_j, \sum_{j=1}^{m} c_{ij} = \lambda_i \right\}. \tag{2.1}$$

This infimum is known as the *Kantorovich distance* in $L(X)$ between $\sum \lambda_i x_i$ and $\sum \mu_j y_j$. It coincides with $||u||$ where $u = \sum \lambda_i x_i - \sum \mu_j y_j \in L^0(X)$ and $||\cdot||$ is the norm defined on $L^0(X)$ as follows. For every $v = \sum_{i=1}^{n} \lambda_i x_i \in L^0(X)$

$$||v|| = \inf \left\{ \sum_{i=1}^{l} |\rho_i| d(a_i, b_i) : v = \sum_{i=1}^{l} \rho_i (a_i - b_i),\ \rho_i \in \mathbb{R}, a_i, b_i \in X \right\}. \tag{2.2}$$

This norm on $L^0(X)$ is called the *Kantorovich norm*, [12]. If (X, d) is a pseudometric space then (2.1) and (2.2) define the *Kantorovich pseudometric* and the *Kantorovich seminorm* respectively.

Let X be a Tychonoff space. Denote by D the family of all continuous pseudometrics on $\overline{X} := X \cup \{\mathbf{0}\}$. For each $d \in D$ there exists a maximal seminorm p_d on $L(X)$ which extends d. We retain the name *Kantorovich seminorm* for p_d (and for its restriction on $L^0(X)$), although several authors use the name *Kantorovich-Rubinstein seminorm*. The vector $\mathbb{R}$-space $L(X)$ and the family of seminorms $\{p_d : d \in D\}$ determine the free locally convex space over X. See Pestov [13], for example, and compare with Raikov [12] in the case of *pointed uniform spaces*. In Section 6 we study the free NA locally convex $\mathbb{F}$-space $L_{\mathbb{F}}(X)$ of an NA uniform space $(X, \mathcal{U})$.

Equation (2.2) has a natural generalization. Let $(\mathbb{F}, |\cdot|)$ be an archimedean valued field and (X, d) be a pseudometric space. For every $v = \sum_{i=1}^{n} \lambda_i x_i \in L^0_{\mathbb{F}}(X)$ define the

Kantorovich seminorm as follows:

$$||v|| = \inf\left\{\sum_{i=1}^{l} |\rho_i| d(a_i, b_i) : v = \sum_{i=1}^{l} \rho_i (a_i - b_i),\ \rho_i \in \mathbb{F}, a_i, b_i \in X\right\}. \tag{2.3}$$

Note that every archimedean valued field $(\mathbb{F}, |\cdot|)$ is essentially a subfield of $\mathbb{C}$ and the valuation is equivalent to the usual valuation on $\mathbb{C}$ (see [16, p. 4] for example).

2.2. **"Democratic" reformulation.** We wish to highlight a point that will become important in the sequel. In the problem described above two disjoint sets $A = \{x_1, \dots, x_n\}$ and $B = \{y_1, \dots, y_m\}$ are considered. The distances between the elements in each set seem irrelevant. Indeed, every distance which appears in Formula (2.1) is between an element of A and an element of B.

Now we consider a more flexible form of the transportation problem (see also [25, p. 44]). Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^{n} \lambda_i = 0$. We have to transfer real numbers between the points $x_1, \dots, x_n \in X$ in the following way. The sum of numbers transferred from x_i minus the sum of numbers transferred to x_i is λ_i. Let c_{ij} denote the real number transferred from point x_i to point x_j. We want to minimize our cost, that is, the value of $\sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ij}| d(x_i, x_j)$. Clearly, one may assume that $c_{ii} = 0$.

As the following lemma suggests, the Kantorovich norm serves both of the approaches described above.

Lemma 2.2 (Democratic reformulation)**.** *If $v = \sum_{i=1}^{n} \lambda_i x_i \in L^0(X)$, then*

$$||v|| = \inf\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ij}| d(x_i, x_j) : \sum_{j=1}^{n} c_{ij} - \sum_{j=1}^{n} c_{ji} = \lambda_i \ \forall i\right\}. \tag{2.4}$$

Proof. Denote by $||v||'$ the expression on the right hand side of Equation (2.4). We want to show that $||v|| = ||v||'$. Let $(c_{ij}) \in \mathbb{R}^{n \times n}$ such that $\sum_{j=1}^{n} c_{ij} - \sum_{j=1}^{n} c_{ji} = \lambda_i$. The coefficient of x_i in $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(x_i - x_j)$ is just $\sum_{j=1}^{n} c_{ij} - \sum_{j=1}^{n} c_{ji}$. It follows that

$$v = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(x_i - x_j).$$

So, by Equation (2.2)

$$||v|| \leq ||v||'.$$

On the other hand, using reductions from [22], we show that if $v = \sum_{i=1}^{l} \rho_i (a_i - b_i)$ then there exists a decomposition $v = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(x_i - x_j)$ with

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ij}| d(x_i, x_j) \leq \sum_{i=1}^{l} |\rho_i| d(a_i, b_i).$$

To see this, first observe that we may assume that $\rho_i > 0\ \forall i$. Consider the following reductions which do not increase the value of the corresponding sum:

(1) Delete any term of v of the form $\rho_i(x - x)$.

(2) If there exist two terms $\lambda(x_i - x_j)$ and $\mu(x_i - x_j)$ with $\lambda, \mu > 0$ replace them with the single term $(\lambda + \mu)(x_i - x_j)$.
(3) Assuming the decomposition contains the term $\lambda(x - z)$ where $z \notin \operatorname{supp}(v)$, then we necessarily have also a term of the form $\mu(z - y)$ where $\lambda, \mu > 0$. We have three subcases to consider replacing in each case the terms $\lambda(x - z)$ and $\mu(z - y)$.
   (a) If $\lambda = \mu$ then replace the pair of terms above with one term $\lambda(x - y)$. It is possible since $\lambda d(x, y) \leq \lambda d(x, z) + \lambda d(z, y)$.
   (b) If $\lambda < \mu$ then replace the terms with $\lambda(x - y)$ and $(\mu - \lambda)(z - y)$. The value of the sum does not increase since

$$\begin{aligned}\lambda d(x,z) + \mu d(z,y) &= \lambda(d(x,z) + d(z,y)) + (\mu - \lambda)d(z,y) \geq \\ &\geq \lambda d(x,y) + (\mu - \lambda)d(z,y).\end{aligned}$$

   (c) If $\lambda > \mu$ then replace the terms with $(\lambda - \mu)(x - z)$ and $\mu(x - y)$. This time we have

$$\begin{aligned}\lambda d(x,z) + \mu d(z,y) &= (\lambda - \mu)d(x,z) + \mu(d(x,z) + d(z,y)) \geq \\ &\geq (\lambda - \mu)d(x,z) + \mu d(x,y).\end{aligned}$$

Using reduction (3) the number of terms containing z decreases. Applying finitely many substitutions of this form and taking into account that the sum of $z's$ coefficients in any decomposition of v is equal to zero, we obtain a decomposition of v with only two terms containing z: $\lambda(x - z)$ and $\lambda(z - y)$. Now use reduction $(3.a)$. Therefore, we can assume that the decomposition only contains terms with support elements. That is, terms of the form $\lambda(x_i - x_j)$ where $\lambda \geq 0$. At this point we use reduction (2) if necessary. We obtain a decomposition $v = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}(x_i - x_j)$ with

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| d(x_i, x_j) \leq \sum_{i=1}^{l} |\rho_i| d(a_i, b_i).$$

It follows that $||v||' \leq ||v||$ and we conclude that $||v|| = ||v||'$. □

Remark 2.3.

(1) Every non-zero element $v \in L^0(X)$ has the form $v = \sum_{i=1}^{n} a_i x_i - \sum_{j=1}^{m} b_j y_j$ where $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$ and $\forall i : 1 \leq i \leq n \ \forall j : 1 \leq j \leq m \ a_i, b_j > 0$. Using this fact one can move back from the democratic approach to the classical one as in Section 2.1.
(2) Using compactness arguments one can prove that the infimum in Formula (2.1) is attained. By the proof of Lemma 2.2, for any minimizing matrix (c_{ij}) from (2.1) there exists a matrix (t_{ij}) from (2.4) such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |t_{ij}| d(x_i, x_j) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| d(x_i, x_j).$$

It follows that the infimum in (2.4) is attained at (t_{ij}).
(3) Replacing $\mathbb{R}$ with $\mathbb{C}$ completely changes the situation. As it follows from [5, 25], in the latter case of $\mathbb{F} = \mathbb{C}$ we cannot even guarantee that the infimum in (2.2) can be approximated by computations on support elements of a vector $u \in L^0_{\mathbb{C}}(X)$. A detailed example is provided in the Appendix (Theorem 8.1 and Example 8.2).

3. Non-archimedean transportation problem

In this section we discuss the main object of our work: a *non-archimedean transportation problem* (NATP). First we recall some definitions.

3.1. **Preliminaries.** A metric space (X, d) is an *ultra-metric space* if d is an *ultra-metric*, i.e., it satisfies *the strong triangle inequality*

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Allowing the distance between distinct elements to be zero we obtain the definition of an *ultra-pseudometric*. It is well known that if $d(x, y) \neq d(y, z)$ then $d(x, z) = \max\{d(x, y), d(y, z)\}$.

A uniform space $(X, \mathcal{U})$ is NA if it has a base B consisting of equivalence relations on X. For every ultra-pseudometric d on X the open balls of radius $\varepsilon > 0$ form a clopen partition of X. So, the uniformity induced by any ultra-pseudometric d on X is NA. A uniformity is NA if and only if it is generated by a system $\{d_i\}_{i \in I}$ of *ultra-pseudometrics*.

Recall that a topological group is *non-archimedean* if it has a base at the identity consisting of open subgroups. For some properties of this class of topological groups see for example [9, 10]. We say that a topological ring (or field or vector space) is NA if its additive group is NA. Note that Lyudkovskii [8] studied NA free Banach spaces.

A *valuation* on a field $\mathbb{F}$ is a function $|\cdot| : \mathbb{F} \to [0, \infty)$ which satisfies the following ($x, y \in \mathbb{F}$):

1. $|x| \geq 0$;
2. $|x| = 0$ if and only if $x = 0_{\mathbb{F}}$;
3. $|x + y| \leq |x| + |y|$;
4. $|xy| = |x||y|$.

Replacing condition (3) with $|x + y| \leq \max\{|x|, |y|\}$ we obtain a *non-archimedean valuation*. In this case the metric d defined by $d(x, y) = |x - y|$ is an ultra-metric.

An (NA) *valued field* is a field $\mathbb{F}$ with a (resp., NA) valuation $|\cdot|$. Every NA valued field is NA as a topological group because every open ball $\{x \in \mathbb{F} : |x| < r\}$ is a (clopen) additive subgroup.

A valuation which is not NA is called an *archimedean valuation*. Let $(\mathbb{F}, |\cdot|)$ be a valued field. A *seminorm* on an $\mathbb{F}$-vector space V is a map $||\cdot|| : V \to [0, \infty)$ such that ($x, y \in V, \alpha \in \mathbb{F}$):

1. $||0_V|| = 0$;
2. $||x + y|| \leq ||x|| + ||y||$;
3. $||\alpha x|| = |\alpha| ||x||$.

If instead of condition (1) we have: $||x|| = 0$ if and only if $x = 0_V$, then $||\cdot||$ is called a *norm*. If the valuation on $\mathbb{F}$ is NA and condition (2) is replaced by $||x + y|| \leq \max\{||x||, ||y||\}$, then the norm (seminorm) $||\cdot||$ is an *ultra-norm* (respectively, ultra-seminorm).

Let $(\mathbb{F}, |\cdot|)$ be an NA valued field. The set $\{|x| : |x| \neq 0\}$ is a subgroup of the multiplicative group $\mathbb{R}_{>0}$ of all positive reals and is said to be the *value group* of the valuation $|\cdot|$. The value group is either discrete or dense in $\mathbb{R}_{>0}$. Accordingly the valuation is called *discrete* or *dense*. If the value group is the trivial subgroup $\{1\}$ then the valuation is said to be *trivial*. For any non-trivial discrete valuation the value group is the infinite cyclic closed subgroup $\{a^k : k \in \mathbb{Z}\}$ of $\mathbb{R}_{>0}$, where $a := \max\{|x| : |x| < 1\}$.

Note that discretely valued fields form a major subclass in the class of NA valued fields. This subclass is closed under taking arbitrary subfields, completions and finite extensions.

The p-adic valuation on the field $\mathbb{Q}$ of rationals is a classical particular case (for every prime p). The completion is the field of p-adic numbers $\mathbb{Q}_p$, a locally compact NA valued field. The valuation of every locally compact NA valued field is discrete (see [16]).

The natural valuation on the field $\mathbb{C}\{\{T\}\}$ of formal Laurent series (which is not locally compact) is discrete [18].

Below we use several times the following well known theorem of Ostrowski (see for example [16, Theorem 1.2]) which shows that the p-adic valuation, up to a natural equivalence, is the only NA non-trivial valuation on $\mathbb{Q}$. In particular, any NA valuation on $\mathbb{Q}$ is discrete.

Theorem 3.1. (Ostrowski's Theorem) *Let $|\cdot|$ be a non-trivial NA valuation on the field $\mathbb{Q}$ of rationals. Then there exists a prime p such that $|\cdot|$ is equivalent to the p-adic valuation $|\cdot|_p$ (namely, there exists $c > 0$ such that $|x| = |x|_p^c \quad \forall x \in \mathbb{Q}$).*

The following is an important example of a densely valued NA field.

Example 3.2. Recall that the elements of the Levi-Civita field $\mathcal{R}$ (see [19] for example) are real functions $f : \mathbb{Q} \to \mathbb{R}$ with left-finite support. That is, for every rational number q the set $A_q := \{a < q |\ f(a) \neq 0\}$ is finite. The field operations are addition and convolution. $\mathbb{R}$ is (algebraically) isomorphic to a subfield of $\mathcal{R}$. Indeed, the map $a \mapsto f_a$ from $\mathbb{R}$ to $\mathcal{R}$, where $f_a(0) = a$ and $f_a(x) = 0\ \forall x \neq 0$, is a field embedding.

For every non-zero element $f \in \mathcal{R}$, the support of f (notation: $\mathrm{supp}(f)$) has a minimum, due to its left-finiteness. Recall that $\mathcal{R}$ admits a natural NA valuation defined by $|f| = e^{-\min \mathrm{supp}(f)}$ for non-zero f. It is easy to see that this valuation is dense. At the same time the restricted valuation on $\mathbb{Q}$ is trivial.

3.2. **Formulation of NATP.** We formulate here a non-archimedean transportation problem using a democratic approach (compare Section 2.2). Let $\mathbb{F}$ be an NA valued field, (X, d) be an ultra-(pseudo)metric space and $x_i \in X$ for every $1 \leq i \leq n$.

We have to transfer field elements between these points in the following way. The sum of elements transferred from x_i minus the sum of elements transferred to x_i is λ_i, where $\lambda_1, \dots, \lambda_n$ are given elements in $\mathbb{F}$ with $\sum\limits_{i=1}^{n} \lambda_i = 0_{\mathbb{F}}$.

Let $c_{ij} \in \mathbb{F}$ denote the element transferred from x_i to x_j. Note that by the setting of NATP we have $\forall i\ \sum\limits_{j=1}^{n} c_{ij} - \sum\limits_{j=1}^{n} c_{ji} = \lambda_i$. We want to minimize as much as possible our max-cost, that is, the value of

$$\max_{1 \leq i,j \leq n} |c_{ij}| d(x_i, x_j).$$

A natural question arises:

Question 3.3. *Is the infimum*

$$\inf \left\{ \max_{1 \leq i,j \leq n} |c_{ij}| d(x_i, x_j) : \forall i\ \sum_{j=1}^{n} c_{ij} - \sum_{j=1}^{n} c_{ji} = \lambda_i \right\} \tag{3.1}$$

attained?

Min-attaining Theorem 5.6 implies that the answer to Question 3.3 is positive for every NA valued field $\mathbb{F}$ (e.g., $\mathbb{Q}_p$) and any ultra-(pseudo)metric space (X, d).

In fact we will show in Theorem 4.3 that (3.1) can be studied via a special ultra-(semi)norm $||\cdot||_d$ on $L_{\mathbb{F}}(X)$. We call it the *Kantorovich ultra-(semi)norm* associated with d (Definition 4.2) because its role is similar to the role of the Kantorovich (semi)norm in the classical transportation problem (with $\mathbb{F} = \mathbb{R}$). Indeed, the infimum in (3.1) coincides with $||u||_d$, where $u = \sum\limits_{i=1}^{n} \lambda_i x_i \in L^0_{\mathbb{F}}(X)$.

4. Kantorovich ultra-norms

Let (X,d) be an ultra-pseudometric space. Consider the set $\overline{X} := X \cup \{\mathbf{0}\}$, where $\mathbf{0} \notin X$. In the sequel we repeatedly use the following simple lemma.

Lemma 4.1. *For every ultra-pseudometric d on X there exists an ultra-pseudometric (denoted also by d) which extends d on $\overline{X} := X \cup \{\mathbf{0}\}$, such that $\mathbf{0}$ is an isolated point in $(\overline{X}, d)$.*

Proof. Fix $x_0 \in X$ and extend the definition of d from X to $\overline{X}$ by letting $d(x,\mathbf{0}) = \max\{d(x,x_0),1\}$. For more details see Claim 1 of [10, Theorem 8.2]. □

Definition 4.2. Let $(\overline{X}, d)$ be an ultra-pseudometric space and $\mathbb{F}$ be an NA valued field. Let us say that an ultra-seminorm p on $L_{\mathbb{F}}(X)$ is *d-compatible* if the pseudometric induced on $\overline{X}$ by p is d. We say that p is a *Kantorovich ultra-seminorm for d* if p is the maximal d-compatible ultra-seminorm on $L_{\mathbb{F}}(X)$.

The maximal property of the Kantorovich norm in the classical non-discrete transportation problem was proved in [12], and this justifies Definition 4.2.

The Kantorovich ultra-norm $||\cdot||$ in Theorem 4.3 serves the NA transportation problem described in Section 3.2. To see this observe that one of the reformulations of this ultra-norm ($m = n$ in Claim 3 below) coincides with the infimum in Formula (3.1) above. Moreover, using the description of the Kantorovich ultra-norm one can obtain an *unbalanced* version of NATP, that is, the case $u = \sum\limits_{i=1}^{n} \lambda_i x_i \notin L^0_{\mathbb{F}}(X)$.

The classical analogue of the following Theorem 4.3 is the Arens-Eells embedding [1]. Its usual verification is based on the dual space, involving the space of Lipschitz functions [5, 25, 6]. In our case the approach is different. If d is a metric on X, then the Kantorovich seminorm defined on $L^0(X)$ is, in fact, a norm. This fact relies on the classical Hahn-Banach theorem (see [25, Corollary 2.2.3]) which does not always hold for general NA Banach spaces, [18, 11]. The proof that the ultra-seminorm in the following theorem is an ultra-norm uses only the fact that the valuation of $\mathbb{F}$ is NA. Below, in Corollary 5.7, we show that the infimum in this theorem is, in fact, a minimum.

Theorem 4.3. (Non-archimedean Arens-Eells embedding)

Let $(\overline{X}, d)$ be an ultra-pseudometric space and $\mathbb{F}$ be an NA valued field.

1. *There exists a Kantorovich ultra-seminorm $||\cdot|| := ||\cdot||_d$ on $L_{\mathbb{F}}(X)$ for d. Furthermore, if d is an ultra-metric then $||\cdot||_d$ is an ultra-norm.*
2. *$||u||$ can be computed on the support of u for every $u \in L_{\mathbb{F}}(X)$. That is,*

$$||u|| = \inf\left\{ \max_{1\le i\le k} |s_i| d(x_i,y_i) : u = \sum_{i=1}^{k} s_i(x_i - y_i),\ x_i, y_i \in \operatorname{supp}(u),\ s_i \in \mathbb{F}\right\}.$$

3. *Moreover, if $u = \sum\limits_{i=1}^{n} \lambda_i x_i$ (normal form) then*

$$||u|| = \inf\left\{ \max_{1\le i,j\le m} |c_{ij}| d(x_i,x_j) : c_{ij} \in \mathbb{F},\ \forall i : 1 \le i \le n\ \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\},$$

where $c_{ii} = 0_{\mathbb{F}}$ and $m = |\operatorname{supp}(u)|$ (see Notation 2.1).

Proof. For $u \in L_{\mathbb{F}}(X)$ define

$$||u|| := \inf\left\{ \max_{1\le i\le n} |\lambda_i| d(x_i,y_i) : u = \sum_{i=1}^{n} \lambda_i(x_i - y_i),\ x_i, y_i \in \overline{X},\ \lambda_i \in \mathbb{F}\right\}.$$

Claim 1: $||\cdot||$ is an ultra-seminorm on $L_{\mathbb{F}}(X)$.

Proof. Clearly, $||u|| \geq 0$ for every $u \in L_{\mathbb{F}}(X)$. Since $\mathbf{0} = \mathbf{0} - \mathbf{0}$ we also have $||\mathbf{0}|| \leq d(\mathbf{0}, \mathbf{0}) = 0$ and thus $||\mathbf{0}|| = 0$. The equality $||\lambda u|| = |\lambda| ||u||$ follows from the fact that for every $\lambda \neq 0_{\mathbb{F}}$, if $u = \sum\limits_{i=1}^{n} \lambda_i(x_i - y_i)$ then $\lambda u = \sum\limits_{i=1}^{n} \lambda\lambda_i(x_i - y_i)$ and, if $\lambda u = \sum\limits_{i=1}^{n} \lambda_i(x_i - y_i)$ then $u = \sum\limits_{i=1}^{n} \lambda^{-1}\lambda_i(x_i - y_i)$. Of course, we also use axiom (4) in the definition of valuation. Finally, observe that

$$||u + v|| \leq \max\{||u||, ||v||\} \quad \forall\, u, v \in L_{\mathbb{F}}(X).$$

Indeed, assuming the contrary, there exist decompositions

$$u = \sum_{i=1}^{k} \lambda_i(x_i - y_i), \;\; v = \sum_{i=k+1}^{l} \lambda_i(x_i - y_i)$$

such that

$$||u + v|| > c := \max\{\max_{1 \leq i \leq k} |\lambda_i| d(x_i, y_i), \max_{k+1 \leq i \leq l} |\lambda_i| d(x_i, y_i)\}.$$

This contradicts the definition of $||u + v||$ since $u + v = \sum\limits_{i=1}^{l} \lambda_i(x_i - y_i)$ with

$$||u + v|| > \max\{\max_{1 \leq i \leq k} |\lambda_i| d(x_i, y_i), \max_{k+1 \leq i \leq l} |\lambda_i| d(x_i, y_i)\} = \max_{1 \leq i \leq l} |\lambda_i| d(x_i, y_i).$$

□

Claim 2: For every $u \in L_{\mathbb{F}}(X)$ the value of $||u||$ can be computed on the support of u. That is,

$$||u|| = \inf\left\{\max_{1 \leq i \leq k} |s_i| d(x_i, y_i) : u = \sum_{i=1}^{k} s_i(x_i - y_i), \;\; x_i, y_i \in \operatorname{supp}(u), \;\; s_i \in \mathbb{F}\right\}.$$

Proof. Let $u = \sum\limits_{i=1}^{k} s_i(x_i - y_i)$ be a decomposition of $u \in L_{\mathbb{F}}(X)$. Consider the following steps which do not increase the value of $\max\limits_{1 \leq i \leq k} |s_i| d(x_i, y_i)$:

(1) Delete any term of u of the form $0_{\mathbb{F}}(x - y)$ or $s_i(x - x)$.
(2) Replace the term $s_i(x_i - y_i)$ with $-s_i(y_i - x_i)$.
(3) Assume there exist $1 \leq i_0 \leq n$ and $\mathbf{0} \neq z \notin \operatorname{supp}(u)$ such that $z = x_{i_0}$ or $z = y_{i_0}$. Using steps $(1) - (2)$ we may assume without loss of generality that the terms $\lambda(x - z)$ and $\mu(z - y)$ appear in the decomposition of $u = \sum\limits_{i=1}^{k} s_i(x_i - y_i)$ with $|\lambda| \leq |\mu|$. Replace them with $\lambda(x - y)$ and $(\mu - \lambda)(z - y)$.

This way the number of terms in which the element z appears decreases. The value of the corresponding maximum $\max\limits_{1 \leq j \leq k} |\mu_j| d(x_j, y_j)$ does not increase under such a substitution, because

$$\max\{|\lambda| d(x, y), |\mu - \lambda| d(z, y)\} \leq \max\{|\lambda| d(x, z), |\mu| d(z, y)\}.$$

Indeed, using the strong triangle inequality and the fact that $|\lambda| \leq |\mu|$ we obtain

$$|\lambda| d(x, y) \leq \max\{|\lambda| d(x, z), |\lambda| d(z, y)\} \leq \max\{|\lambda| d(x, z), |\mu| d(z, y)\}.$$

Also, assuming that $|\mu - \lambda| d(z, y) > |\mu| d(z, y)$ we obtain $|\mu - \lambda| > \max\{|\lambda|, |\mu|\}$, which contradicts the strong triangle inequality. Thus, $|\mu - \lambda| d(z, y) \leq |\mu| d(z, y)$. Applying finitely many substitutions of this form and taking into account that

the sum of $z's$ coefficients in any decomposition of u is equal to zero, we obtain a decomposition of u with only two terms in which z appears: $\lambda(x-z)$ and $\lambda(z-y)$. These terms can be replaced by the single term $\lambda(x-y)$ since $\lambda(x-z)+\lambda(z-y) = \lambda(x-y)$ and $|\lambda|d(x,y) \leq \max\{|\lambda|d(x,z), |\lambda|d(z,y)\}$. Now the term $\lambda(x-y)$ and all other terms in the new decomposition do not contain the element z.

(4) Assume there exist $1 \leq i_0 \leq n$ and $z = \mathbf{0} \notin \operatorname{supp}(u)$ such that $z = x_{i_0}$ or $z = y_{i_0}$. We claim that similar to case (3) it suffices to consider decompositions $u = \sum\limits_{i=1}^{k} s_i(x_i - y_i)$ that contain terms of the form $\lambda(x-z)$ and $\mu(z-y)$ with $|\lambda| \leq |\mu|$. Indeed, since $z = \mathbf{0} \notin \operatorname{supp}(u)$ it follows that $u = \sum\limits_{i=1}^{n} \lambda_i t_i \in L^0_{\mathbb{F}}(X)$ (normal form) and $\sum\limits_{i=1}^{n} \lambda_i = 0_{\mathbb{F}}$. If there exists only one term $\lambda(x-z)$ in which z appears then by the previous steps we can assume, without loss of generality, that we are dealing with a decomposition of u of the form $u = \sum\limits_{j=1}^{k} \mu_j(a_j - b_j) + \lambda(t_1 - z)$, where $a_j, b_j \in \operatorname{supp}(u)$. On the one hand, $u' = \sum\limits_{j=1}^{k} \mu_j(a_j - b_j) \in L^0_{\mathbb{F}}(X)$. On the other hand, for every $i \neq 1$ the sum of the coefficients of t_i in this decomposition of u' is equal to λ_i. It follows that the sum of $t_1's$ coefficients in u' is λ_1 and this implies that $\lambda = 0_{\mathbb{F}}$. So we may assume that the terms $\lambda(x-z)$ and $\mu(z-y)$ appear in the decomposition of $u = \sum\limits_{i=1}^{k} s_i(x_i - y_i)$ with $|\lambda| \leq |\mu|$. If $\lambda = \mu$ we simply replace these terms with the single term $\lambda(x-y)$. Otherwise, replace these terms with $\lambda(x-y)$ and $(\mu-\lambda)(z-y)$. In any case we can say that completely similar to reduction (3) the number of terms in which the element z appears decreases. The value of the corresponding maximum $\max\limits_{1 \leq j \leq k} |\mu_j| d(x_j, y_j)$ does not increase under such a substitution. We apply finitely many substitutions of this form and obtain a decomposition of u in which all terms do not contain the element z.

Using reductions (3) and (4) we complete the proof of Claim 2. $\square$

Claim 3: For $u = \sum\limits_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$ let $m = |\operatorname{supp}(u)|$ (by Notation 2.1 we have $m = n$, or $m = n+1$). Then,

$$||u|| = \inf\left\{ \max_{1 \leq i,j \leq m} |c_{ij}| d(x_i, x_j) : c_{ij} \in \mathbb{F},\ \forall i : 1 \leq i \leq n \ \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\}, \tag{4.1}$$

where $c_{ii} = 0_{\mathbb{F}}$.

Proof. By Notation 2.1, $\sum\limits_{i=1}^{n} \lambda_i x_i$ is a normal form of u. It follows that a matrix $(c_{ij}) \in \mathbb{F}^{m \times m}$ satisfies the equations

$$\sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \ \forall i : 1 \leq i \leq n$$

if and only if $u = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}(x_i - x_j)$. Indeed, on the one hand the coefficient of x_i in the right expression is $\sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji}$ for all $1 \leq i \leq n$. On the other hand, the coefficient of x_i in u is λ_i. Note that by our convention if $m = n+1$ then $x_{n+1} = \mathbf{0}$. Since $d(x_i, x_i) = 0$ and $c_{ii} - c_{ii} = 0_{\mathbb{F}}$ we may assume without loss of generality that $c_{ii} = 0_{\mathbb{F}}$.

By Claim 2, $||u||$ can be computed on the support of u. If we have two terms of the form $\lambda(x_i - x_j)$, $\mu(x_i - x_j)$ we can replace them with the single term $(\lambda + \mu)(x_i - x_j)$ since $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$. Thus, we may consider only decompositions of the form $u = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(x_i - x_j)$, where $\sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i$ and $c_{ii} = 0_{\mathbb{F}}$. □

Claim 4: For $u = \sum_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$ let $m = |\operatorname{supp}(u)|$. Then,

$$||u|| \geq r \cdot l_0$$

where $r = \max\{|\lambda_i| : 1 \leq i \leq n\}$ and $l_0 = \min\{d(x_i, x_j) : \ 1 \leq i \neq j \leq m\}$.

Proof. Assuming the contrary let $||u|| < r \cdot l_0$. By Claim 3 there exists a matrix $(c_{ij}) \in \mathbb{F}^{m\times m}$ such that $\sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \ \forall i : 1 \leq i \leq n$ and in addition $r \cdot l_0 > \max_{1\leq i,j\leq m} |c_{ij}| d(x_i, x_j)$. Taking into account the definition of l_0 we get $r > |c_{ij}| \ \forall i, j$. By the definition of r there exists $1 \leq i_0 \leq n$ such that $r = |\lambda_{i_0}|$. Thus, $|\lambda_{i_0}| > |c_{ij}| \ \forall i, j$. In particular,

$$|\lambda_{i_0}| > \max\{\max_{1\leq j\leq m} |c_{i_0 j}|, \max_{1\leq j\leq m} |c_{j i_0}|\}.$$

Applying the strong triangle inequality to the equation $\sum_{j=1}^{m} c_{i_0 j} - \sum_{j=1}^{m} c_{j i_0} = \lambda_{i_0}$ we obtain the contradiction

$$|\lambda_{i_0}| \leq \max\{\max_{1\leq j\leq m} |c_{i_0 j}|, \max_{1\leq j\leq m} |c_{j i_0}|\}.$$

□

Claim 5: $\iota : (\overline{X}, d) \hookrightarrow (L_{\mathbb{F}}(X), || \cdot ||)$, $\iota(x) = \{x\}$ is an isometric embedding, i.e.

$$||x - y|| = d(x, y) \quad \forall\ x, y \in \overline{X}.$$

Proof. If $x = y$ the assertion is trivial so we may assume that $u = x - y \neq \mathbf{0}$. By Claim 2 the value $||x-y||$ can be computed on the support $\{x, y\}$. Using also some of the reductions we mentioned above, it suffices to consider only the trivial decomposition $u = x - y$. It follows that $||x - y|| = d(x, y)$. □

Claim 6: $||u|| = 0$ if and only if u admits a presentation $u = \sum_{k=1}^{t} s_k(x_k - y_k)$ such that $x_k, y_k \in \operatorname{supp}(u)$ and $d(x_k, y_k) = 0$ for every $k \in \{1, \ldots, t\}$. In particular, the ultra-seminorm $|| \cdot ||$ is an ultra-norm on $L_{\mathbb{F}}(X)$ if and only if d is an ultra-metric on X.

Proof. The "if" part is trivial.

The "only if" part is obvious for $u = \mathbf{0}$. Suppose that $u \neq \mathbf{0}$ and let $u = \sum_{i=1}^{n} \lambda_i x_i$ be a normal form of u. First suppose that u is *d-irreducible* in the following sense: there are no $1 \leq i \neq j \leq m$ such that $d(x_i, x_j) = 0$, where $m = |\operatorname{supp}(u)|$. We claim that $||u|| > 0$. Indeed, the corresponding l_0 defined in Claim 4 is positive and we have $||u|| \geq r \cdot l_0$. Clearly, $r > 0$ because $u \neq \mathbf{0}$. So, we get that $||u|| > 0$.

Now we can suppose that u is d-reducible. We describe a certain reduction for u. Choose a pair $i \neq j$ such that $d(x_i, x_j) = 0$. Without loss of generality we may assume that $x_i \neq \mathbf{0}$. Denote $w_1 := \lambda_i(x_i - x_j), u_1 := u - w_1$. By Claims 1 and 5 we know that $||w_1|| = 0$. Hence, $||u|| = ||u - w_1|| = 0$.

▶ In case $x_j = \mathbf{0}$ delete the term $\lambda_i x_i$ in the presentation $u = \sum_{i=1}^{n} \lambda_i x_i$ to obtain a normal form of $u - w_1$.

▶ In case $x_j \neq \mathbf{0}$ observe that $\lambda_i x_i + \lambda_j x_j = \lambda_i(x_i - x_j) + (\lambda_i + \lambda_j)x_j$. Replacing the terms $\lambda_i x_i, \lambda_j x_j$ in the presentation $u = \sum_{i=1}^{n} \lambda_i x_i$ with the single term $(\lambda_i + \lambda_j)x_j$ we get a normal form of $u - w_1$.

In both cases we can then use the same reductions for u_1 to obtain $u_2 := u_1 - w_2$, etc. Continuing in this manner we get, after finitely many steps, a vector u_t such that $||u|| = ||u_t|| = 0$ and in the normal presentation of u_t we have no pair of distinct elements $a, b \in X$ such that $d(a, b) = 0$. That is, u_t is d-irreducible. Then necessarily $u_t = \mathbf{0}$. Indeed, if not, then as above we obtain that $||u_t|| > 0$.

So, $u_t = \mathbf{0}$. Hence, $u = \sum_{k=1}^{t} w_k$. By the definition of w_k this proves Claim 6. □

Claim 7: (Maximality property) Let σ be an ultra-seminorm on $L_{\mathbb{F}}(X)$ such that

$$\sigma(x - y) \leq d(x, y) \quad \forall x, y \in \overline{X}. \tag{4.2}$$

Then $\sigma \leq || \cdot ||$.

Proof. Let u be a non-zero element of $L_{\mathbb{F}}(X)$ and σ be an ultra-seminorm which satisfies (4.2). Then for every decomposition $u = \sum_{i=1}^{n} \lambda_i(x_i - y_i),\ x_i, y_i \in \overline{X}$ we obtain

$$\sigma(u) = \sigma(\sum_{i=1}^{n} \lambda_i(x_i - y_i)) \leq \max_{1 \leq i \leq n} |\lambda_i| \sigma(x_i - y_i) \leq \max_{1 \leq i \leq n} |\lambda_i| d(x_i, y_i).$$

It follows from the definition of the ultra-seminorm $|| \cdot ||$ that $\sigma(u) \leq ||u||$. □

Combining the claims we complete the proof of Theorem 4.3. □

Example 4.4. Let $\mathbb{F} := \mathbb{Z}_2$ be the discrete field of two elements. Note that in this case $(L_{\mathbb{F}}(X), || \cdot ||)$, as a topological group, coincides with $B_{\mathcal{NA}}$ the *uniform free NA Boolean group* over (X, d). Indeed, this follows from the fact that $B_{\mathcal{NA}}$ is metrizable by a Graev type ultra-norm (see [10]).

Remark 4.5. Theorem 5.6 shows that in Theorem 4.3 we can assume, in addition, that:

(1) The infimum in Theorem 4.3 is attained.
(2) The coefficients c_{ij} (in Theorem 4.3.3) belong to the additive subgroup G_u of $\mathbb{F}$, generated by the normal coefficients λ_i of u.

Remark 4.6. Using Claim 3 and additional computations we obtain a simplified version of Equation (4.1):

$$||u|| = \min \left\{ \max_{1 \leq i < j \leq m} |c_{ij}| d(x_i, x_j) : \forall i \geq j\ c_{ij} = 0\ ,\ \forall i : 1 \leq i \leq n\ \sum_{j=i+1}^{m} c_{ij} - \sum_{j=1}^{i-1} c_{ji} = \lambda_i \right\}.$$

5. Generalized integer value property

5.1. ***G*-value property for subgroups $G \subseteq \mathbb{R}$.** First recall the *integer value property* for the case $\mathbb{F} = \mathbb{R}$. Let d be a (pseudo)metric on X and $||\cdot||$ be its Kantorovich (semi)norm. For an element of $L^0(X)$ with integer coefficients the inf-sum cost Formula (2.1) achieves its infimum at an integer matrix (c_{ij}). See, for example, Sakarovitz [17, p. 179], and Uspenskij [22].

Replacing the group of integers $\mathbb{Z}$ with any other additive subgroup G of $\mathbb{R}$ we obtain a natural generalization. We call it the *G-value property.* It means that whenever we have an element of $L^0(X)$ with coefficients from G, the minimum in the formula is obtained at a matrix with elements from G. This generalized version can be proved using the tools of convex analysis as in [22].

In the sequel we prove the G-value property for the NA case.

5.2. ***G*-value property in the non-archimedean case.** In this subsection let $\mathbb{F}$ be an NA valued field and (X, d) be an ultra-(pseudo)metric space.

Lemma 5.1. *Let G be an additive subgroup of an NA valued field $\mathbb{F}$. Let $u = \sum_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$ with $\lambda_i \in G$ $\forall i$. Then the ultra-seminorm $||u||$ can be computed using only the coefficients from G. That is, in the formula of Theorem 4.3.2 we get*

$$||u|| := \inf \left\{ \max_{1 \le k \le l} |\rho_k| d(s_k, t_k) : u = \sum_{k=1}^{l} \rho_k (s_k - t_k),\ s_k, t_k \in \overline{X},\ \rho_k \in G \right\}.$$

Proof. It is equivalent to show that for every decomposition $u = \sum_{j=1}^{m} \mu_j (a_j - b_j)$ there exists a decomposition $u = \sum_{k=1}^{l} \rho_k (s_k - t_k)$ with $\rho_k \in G$ $\forall k : 1 \le k \le l$ such that

$$\max_{1 \le k \le l} |\rho_k| d(s_k, t_k) \le \max_{1 \le j \le m} |\mu_j| d(a_j, b_j).$$

By deleting any term of u of the form $\mu_j(x - x)$ we may assume that $a_j \neq b_j$ $\forall j$. If $\mu_j \in G$ $\forall j : 1 \le j \le m$ there is nothing to prove. So, without loss of generality, we may assume that $\mu_1 \notin G$.

Moreover we can suppose that $a_1 \neq \mathbf{0}$ (otherwise, write the summand $(-\mu_1)(b_1 - a_1)$ instead of $\mu_1(a_1 - b_1)$). Consider the set of indices

$$A := \{j \neq 1 : a_j = a_1 \vee b_j = a_1\}.$$

We show that there exists $j \in A$ such that $\mu_j \notin G$. If $a_1 \in \operatorname{supp}(u)$ then there exists $1 \le i \le n$ such that $a_1 = x_i$. Hence,

$$\mu_1 + \sum_{j \in A} k_j \mu_j = \lambda_i$$

where $k_j = 1$ if $a_j = a_1$ and $k_j = -1$ if $b_j = a_1$. If $a_1 \notin \operatorname{supp}(u)$ then

$$\mu_1 + \sum_{j \in A} k_j \mu_j = 0_{\mathbb{F}}.$$

Since G is an additive subgroup of $\mathbb{F}$, $\mu_1 \notin G$ and $\{0_{\mathbb{F}}, \lambda_i\} \subseteq G$, we conclude that there exists $j \in A$ such that $\mu_j \notin G$.

Since $|\mu_j| = |-\mu_j|, |\mu_1| = |-\mu_1|$ we may assume, without loss of generality, that there exists $j \neq 1$ such that $b_j = a_1$, $\mu_j \notin G$ and $|\mu_j| \le |\mu_1|$. Replace the terms $\mu_1(a_1 - b_1)$ and

$\mu_j(a_j - a_1)$ with $\mu_j(a_j - b_1)$ and $(\mu_1 - \mu_j)(a_1 - b_1)$. We show that

$$\max\{|\mu_j|d(a_j, b_1), |\mu_1 - \mu_j|d(a_1, b_1)\} \le \max\{|\mu_j|d(a_j, a_1), |\mu_1|d(a_1, b_1)\}.$$

This way we decrease the number of terms in which the element a_1 appears with scalar coefficient not from G. Since $|\mu_j| \le |\mu_1|$ it follows from the strong triangle inequality of the valuation $|\cdot|$ that

$$|\mu_1 - \mu_j|d(a_1, b_1) \le \max\{|\mu_1|d(a_1, b_1), |\mu_j|d(a_1, b_1)\} = |\mu_1|d(a_1, b_1) \le$$
$$\le \max\{|\mu_j|d(a_j, a_1), |\mu_1|d(a_1, b_1)\}.$$

From the strong triangle inequality of d we obtain

$$|\mu_j|d(a_j, b_1) \le \max\{|\mu_j|d(a_j, a_1), |\mu_j|d(a_1, b_1)\} \le \max\{|\mu_j|d(a_j, a_1), |\mu_1|d(a_1, b_1)\}.$$

Therefore,

$$\max\{|\mu_j|d(a_j, b_1), |\mu_1 - \mu_j|d(a_1, b_1)\} \le \max\{|\mu_j|d(a_j, a_1), |\mu_1|d(a_1, b_1)\}.$$

Applying finitely many substitutions of this form to terms in which the element a_1 appears and in which the coefficients are not taken from G, we obtain a decomposition in which all coefficients of a_1 (if there are any) are from G. Repeating this algorithm for other elements, if necessary, we obtain a decomposition of the form

$$u = \sum_{k=1}^{l} \rho_k(s_k - t_k)$$

with $\rho_k \in G\ \forall k : 1 \le k \le l$ such that

$$\max_{1\le k\le l} |\rho_k|d(s_k, t_k) \le \max_{1\le j\le m} |\mu_j|d(a_j, b_j).$$

□

Notation 5.2. For every $u = \sum\limits_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$ (normal form) denote by G_u the additive subgroup of $\mathbb{F}$ generated by the coefficients λ_i of u.

Observe that by the strong triangle inequality for every $c \in G_u$ we have

$$|c| \le r := \max\{|\lambda_i| : 1 \le i \le n\}. \tag{5.1}$$

Lemma 5.3. (NA local G_u-value property) *For every $u = \sum\limits_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$ we have*

$$||u|| = \inf\left\{ \max_{1\le i,j\le m} |c_{ij}|d(x_i, x_j) : c_{ij} \in G_u, \quad \forall i : 1 \le i \le n \ \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\}.$$

Proof. Combine Lemma 5.1 with Claims 2, 3 of Theorem 4.3 taking into account the following observation. Let $u = \sum\limits_{k=1}^{l} \rho_k(s_k - t_k)$ with $\rho_k \in G\ \forall k : 1 \le k \le l$. Since G is an additive subgroup of $\mathbb{F}$, each reduction appearing in the proof of Claim 2 yields a decomposition of the same form. That is, the coefficients in the resulting decomposition are from G. □

Lemma 5.4. *Let $u = \sum\limits_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$. Suppose that for every positive reals $a \le b$ the set $A_{ab} := \{|x| : x \in G_u,\ a \le |x| \le b\}$ is finite. Then*

$$||u|| = \min\left\{ \max_{1\le i,j\le m} |c_{ij}|d(x_i, x_j) : c_{ij} \in G_u, \forall i : 1 \le i \le n \ \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\}.$$

Proof. In case $||u|| = 0$ we do not need the finiteness assumption. Indeed, by the proof of Claim 6 of Theorem 4.3 there exists a matrix $(c_{ij}) \in G_u^{m\times m}$ such that

$$u = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(x_i - x_j),$$

where for every i, j either $d(x_i, x_j) = 0$ or $c_{ij} = 0_{\mathbb{F}}$. It follows that

$$\sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \ \forall i : 1 \le i \le n$$

and thus the infimum in Lemma 5.3 is attained. So without restriction of generality we may assume that $||u|| > 0$. We have to show that the infimum in Lemma 5.3 is attained. Assuming the contrary and taking into account Formula (5.1), there exists a sequence of matrices

$$\{(c_{ij}^k) : k \in \mathbb{N}\} \subseteq G_u^{m\times m}$$

with the following properties:

(1) $\forall i, j, k \quad |c_{ij}^k| \le r$;

(2) $\forall k \in \mathbb{N}\ \forall i : 1 \le i \le n \quad \sum\limits_{j=1}^{m} c_{ij}^k - \sum\limits_{j=1}^{m} c_{ji}^k = \lambda_i$;

(3) $\max\limits_{1\le i,j\le m} |c_{ij}^k| d(x_i, x_j) > \max\limits_{1\le i,j\le m} |c_{ij}^{k+1}| d(x_i, x_j) > ||u||$.

Passing to a subsequence, if necessary, we can also assume that there exists a pair of indices (i_0, j_0) such that

$$\forall k \in \mathbb{N} \ \max_{1\le i,j\le m} |c_{ij}^k| d(x_i, x_j) = |c_{i_0 j_0}^k| d(x_{i_0}, x_{j_0}).$$

It follows that

$$\forall k \in \mathbb{N} \quad r \ge |c_{i_0 j_0}^k| > |c_{i_0 j_0}^{k+1}| > \frac{||u||}{d(x_{i_0}, x_{j_0})} > 0.$$

By our assumption the set

$$A = \left\{ |x| : x \in G_u,\ r \ge |x| \ge \frac{||u||}{d(x_{i_0}, x_{j_0})} \right\}$$

is finite. This contradicts the fact that the set $\{|c_{i_0 j_0}^k| : k \in \mathbb{N}\}$, being a strictly decreasing sequence, is infinite. □

By $\mathrm{char}(\mathbb{F})$ we denote the characteristic of the field $\mathbb{F}$. Recall that if $\mathrm{char}(\mathbb{F}) = 0$ then the field $\mathbb{Q}$ of rationals is naturally embedded in $\mathbb{F}$.

Lemma 5.5. *Let* $(\mathbb{F}, |\cdot|)$ *be an NA valued field with* $\mathrm{char}(\mathbb{F}) = 0$. *Then, for every positive reals* $a \le b$ *the set* $\{|q| :\ a \le |q| \le b,\ q \in \mathbb{Q}\}$ *is finite.*

Proof. By Ostrowski's Theorem 3.1 the restricted valuation on $\mathbb{Q} \subseteq \mathbb{F}$ is discrete. Hence, the set $\{|q| : q \in \mathbb{Q} \setminus \{0_{\mathbb{F}}\}\}$ is closed and discrete. It follows that for any positive reals $a \le b$ the set $\{|q| :\ a \le |q| \le b,\ q \in \mathbb{Q}\}$ is compact and discrete and thus finite. □

Theorem 5.6 (Min-attaining Theorem)**.** *Let* $(\mathbb{F}, |\cdot|)$ *be an NA valued field. Let* $u = \sum\limits_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$. *Then,*

$$||u|| = \min\left\{ \max_{1\le i,j\le m} |c_{ij}| d(x_i, x_j) : c_{ij} \in G_u,\ \forall i : 1 \le i \le n \ \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\}.$$

Proof. We show that Lemma 5.4 can be applied to every NA valued field $(\mathbb{F}, |\cdot|)$ and to every $u = \sum_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$.

► In case $\mathrm{char}(\mathbb{F}) > 0$ the subgroup G_u is finite, being a finitely generated additive subgroup of a field of positive characteristic. So, it is trivial that the set A_{ab} from Theorem 5.4 is finite.

► Now assume that $\mathrm{char}(\mathbb{F}) = 0$. Instead of showing directly that the set

$$A_{ab} := \{|x| : x \in G_u,\ a \leq |x| \leq b\}$$

is finite for every positive reals $a \leq b$, we will show that it is contained in a finite subset B_{ab} of $\mathbb{R}$. Let

$$B_{ab} := \{|x| : x \in \widetilde{G_u},\ a \leq |x| \leq b\}$$

where $\widetilde{G_u} := \{\sum_{i=1}^{n} m_i\lambda_i |\ m_i \in \mathbb{Q}\}$. Since $G_u \subseteq \widetilde{G_u}$ we also have $A_{ab} \subseteq B_{ab}$. We prove the finiteness of the set B_{ab} using induction on n, the number of scalar coefficients λ_i in the normal form of u.

First, for the case $n = 1$ let $u = \lambda x$. We show that the set $\{|m\lambda| : a \leq |m\lambda| \leq b,\ m \in \mathbb{Q}\}$ is finite. It is equivalent to show that the set $\{|m| : c \leq |m| \leq d,\ m \in \mathbb{Q}\}$ is finite, where $c = \frac{a}{|\lambda|}, d = \frac{b}{|\lambda|}$. This set is finite by Lemma 5.5.

Let $u = \sum_{i=1}^{n+1} \lambda_i x_i$ and $v = \sum_{i=1}^{n} \lambda_i x_i$. By the induction hypothesis the set

$$C_{ab} := \{|x| : x \in \widetilde{G_v},\ a \leq |x| \leq b\}$$

is finite. If

$$\{|x| : x \in \widetilde{G_u} \setminus \widetilde{G_v},\ a \leq |x| \leq b\} = \emptyset$$

there is nothing to prove.

So we may assume that there exists an element of $\widetilde{G_u}$ of the form $t = \sum_{i=1}^{n+1} t_i\lambda_i$, where $t_i \in \mathbb{Q}\ \forall i,\ \ t_{n+1} \neq 0,\ a \leq |t| \leq b$ and $|t| \notin C_{ab}$. It follows from Lemma 5.5 that the set

$$D := \{|qt| : q \in \mathbb{Q},\ a \leq |qt| \leq b\}$$

is finite. It suffices to show that

$$\{|x| : x \in \widetilde{G_u} \setminus \widetilde{G_v},\ a \leq |x| \leq b\} \subseteq C_{ab} \cup D.$$

Let $s = \sum_{i=1}^{n+1} s_i\lambda_i \in \widetilde{G_u} \setminus \widetilde{G_v}$ and $a \leq |s| \leq b$.

We will show that $|s| \in C_{ab} \cup D$. Since $s \in \widetilde{G_u} \setminus \widetilde{G_v}$ then $s_{n+1} \neq 0$. Since $t_{n+1} \neq 0$, it follows that $\exists q \in \mathbb{Q} \setminus \{0\}$ such that $qt_{n+1} = s_{n+1}$. Thus there exists $r \in \widetilde{G_v}$ such that $s = qt + r$. Clearly $|qt| \neq |r|$. Indeed, otherwise, we have $|t| = |\frac{1}{q}r|$ contradicting the fact that $|t| \notin C_{ab}$. So, by the basic properties of the strong triangle inequality, either $|s| = |qt| \in D$ or $|s| = |r| \in C_{ab}$. Therefore $B_{ab} \subseteq C_{ab} \cup D$, as needed. □

Corollary 5.7. *The infimum in Theorem 4.3 is, in fact, a minimum.*

Proposition 5.8. *For every $u = \sum_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$ we have*

$$r \cdot l_0 \leq ||u|| \leq r \cdot l_1$$

where $r = \max\{|\lambda_i| : 1 \leq i \leq n\}$, $l_1 = \max\{d(x_i, x_j) :\ 1 \leq i, j \leq m\}$, $l_0 = \min\{d(x_i, x_j) : 1 \leq i \neq j \leq m\}$ and $m = |\,\mathrm{supp}(u)|$.

Proof. Claim 4 of Theorem 4.3 provides a lower bound $r \cdot l_0 \leq ||u||$.

By Theorem 5.3

$$||u|| = \inf\left\{ \max_{1\leq i,j\leq m} |c_{ij}| d(x_i, x_j) : c_{ij} \in G_u, \forall i : 1 \leq i \leq n \ \sum_{j=1}^{n} c_{ij} - \sum_{j=1}^{n} c_{ji} = \lambda_i \right\},$$

while $|c_{ij}| \leq r$ by (5.1). Therefore $||u|| \leq \max_{1\leq i,j\leq m} |c_{ij}| d(x_i, x_j) \leq r \cdot l_1$. $\square$

Corollary 5.9. *Let $u = \sum_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)$. Suppose that $l = d(x_i, x_j)$ for every $x_i \neq x_j \in$ $\mathrm{supp}(u)$. Then $||u|| = r \cdot l$ where $r = \max\{|\lambda_i| : 1 \leq i \leq n\}$.*

6. Free NA locally convex space

For the free locally convex $\mathbb{F}$-spaces (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) on uniform spaces we refer to Raikov [12]. Here we consider their NA analogue. Let $\mathbb{F}$ be an NA valued field. Recall [18, 11] that a Hausdorff NA $\mathbb{F}$-vector space V is said to be *locally convex* if its topology can be generated by a family of ultra-seminorms.

Assigning to every NA locally convex $\mathbb{F}$-space V its uniform space $(V,\mathcal{U})$, we define a forgetful functor from the category ${}_{\mathbb{F}}\mathrm{LCS}_{\mathcal{NA}}$ of all Hausdorff NA locally convex spaces to the category of all NA Hausdorff uniform spaces $\mathbf{Unif}_{\mathcal{NA}}$.

Definition 6.1. Let $\mathbb{F}$ be an NA valued field and $(X,\mathcal{U}) \in \mathbf{Unif}_{\mathcal{NA}}$ be an NA uniform space. By a *free NA locally convex $\mathbb{F}$-space* of $(X,\mathcal{U})$ we mean a pair $(L_{\mathbb{F}}(X,\mathcal{U}), i)$ (or, simply, $L_{\mathbb{F}}(X,\mathcal{U})$ or $L_{\mathbb{F}}(X)$ when i and $\mathcal{U}$ are understood), where $L_{\mathbb{F}}(X,\mathcal{U})$ is a locally convex $\mathbb{F}$-space and $i : X \to L_{\mathbb{F}}(X,\mathcal{U})$ is a uniform map satisfying the following universal property. For every uniformly continuous map $\varphi : (X,\mathcal{U}) \to V$ into a locally convex $\mathbb{F}$-space V, there exists a unique continuous linear homomorphism $\Phi : L_{\mathbb{F}}(X,\mathcal{U}) \to V$ for which the following diagram commutes:

$$\begin{tikzcd}
(X,\mathcal{U}) \arrow[r, "i"] \arrow[dr, "\varphi"'] & L_{\mathbb{F}}(X,\mathcal{U}) \arrow[d, "\Phi"] \\
& V
\end{tikzcd}$$

A categorical reformulation of this definition is that $i : X \to L_{\mathbb{F}}(X,\mathcal{U})$ is a universal arrow from $(X,\mathcal{U})$ to the forgetful functor ${}_{\mathbb{F}}\mathrm{LCS}_{\mathcal{NA}} \to \mathbf{Unif}_{\mathcal{NA}}$. The uniformity $\overline{\mathcal{U}}$ in the following theorem is obtained from the uniformity $\mathcal{U}$ by adding to X the element $\mathbf{0}$ as an isolated point. In particular, if $\mathcal{U}$ is metrizable and d is the corresponding ultra-metric, one can extend d from X to $\overline{X}$ such that d induces the uniformity $\overline{\mathcal{U}}$ (apply Lemma 4.1).

Theorem 6.2. *For every Hausdorff NA uniform space $(X,\mathcal{U})$ the uniform NA free locally convex $\mathbb{F}$-space exists. Its structure can be defined as follows. Let D be the set of all $\overline{\mathcal{U}}$-uniformly continuous ultra-pseudometrics on $\overline{X} := X \cup \{\mathbf{0}\}$. For every $d \in D$ we have the corresponding Kantorovich ultra-seminorm $||\cdot||_d$ on $L_{\mathbb{F}}(X)$. Then $L_{\mathbb{F}}(X)$ endowed with the family $\Gamma := \{||\cdot||_d : d \in D\}$ of Kantorovich ultra-seminorms defines the desired uniform NA free locally convex $\mathbb{F}$-space which we denote by $L_{\mathbb{F}}(X,\mathcal{U})$. The corresponding arrow $i : (X,\mathcal{U}) \to L_{\mathbb{F}}(X,\mathcal{U})$ is a uniform embedding.*

Proof. First of all, observe that $L_{\mathbb{F}}(X,\mathcal{U})$ is Hausdorff. Indeed, this follows by analyzing Claims 4 and 6 of Theorem 4.3 (or, Proposition 5.8).

Next we have the following commutative diagram

$$\begin{tikzcd}
(X,\mathcal{U}) \arrow[r, "i"] \arrow[dr, "\varphi"'] & L_{\mathbb{F}}(X,\mathcal{U}) \arrow[d, "\Phi"] \\
 & V
\end{tikzcd}$$

Now we only have to show that Φ is continuous. Since $\mathbf{0}$ is isolated in $(\overline{X},\overline{\mathcal{U}})$ and $\varphi : (X,\mathcal{U}) \to V$ is uniformly continuous, so is the natural extension $\varphi : (\overline{X},\overline{\mathcal{U}}) \to V$. By our assumption V has a family Γ_V of ultra-seminorms which generate its topology. Every $\rho \in \Gamma_V$ induces an ultra-seminorm σ_ρ on $L_{\mathbb{F}}(X)$ and an ultra-pseudometric d_ρ on $\overline{X}$ defined by

$$\sigma_\rho(u) := \rho(\Phi(u)), \quad d_\rho(x,y) := \rho(\varphi(x) - \varphi(y)),$$

respectively.

Since $\varphi : (\overline{X},\overline{\mathcal{U}}) \to V$ is uniformly continuous we have $d_\rho \in D$. Consider the corresponding Kantorovich ultra-seminorm $||\cdot||_{d_\rho}$ on $L_{\mathbb{F}}(X)$. Then $\sigma_\rho(x-y) = d_\rho(x,y)$ for every $x,y \in \overline{X}$. By the maximality property (Definition 4.2 and Theorem 4.3) we obtain $||\cdot||_{d_\rho} \geq \sigma_\rho$. This guarantees that $\rho(\Phi(u)) \leq ||u||_{d_\rho}$ for every $u \in L_{\mathbb{F}}(X)$, which implies the continuity of Φ.

Finally, note that by Lemma 4.1 and Theorem 4.3 the family Γ of Kantorovich ultra-seminorms generates the original uniform structure $\mathcal{U}$ on $X = i(X) \subseteq L_{\mathbb{F}}(X)$. Hence i is a uniform embedding. □

Proposition 6.3. *Let $\mathbb{F}$ be an NA valued filed and K a subfield of $\mathbb{F}$. Then for every Hausdorff NA uniform space $(X,\mathcal{U})$ the natural algebraic inclusion $j : L_K(X) \to L_{\mathbb{F}}(X)$ of K-vector spaces is a topological embedding.*

Proof. Let d be a uniformly continuous ultra-pseudometric on $\overline{X} := X \cup \{\mathbf{0}\}$. Denote by $||\cdot||^K$ and $||\cdot||^{\mathbb{F}}$ the corresponding Kantorovich ultra-seminorms of d in $L_K(X)$ and $L_{\mathbb{F}}(X)$ respectively. Let $u = \sum\limits_{i=1}^{n} \lambda_i x_i \in L_K(X) \subseteq L_{\mathbb{F}}(X)$. Then clearly G_u is an additive subgroup of K and of $\mathbb{F}$. Therefore by Theorem 5.3 we have $||u||^K = ||u||^{\mathbb{F}}$. Now Theorem 6.2 guarantees that $j : L_K(X) \to L_{\mathbb{F}}(X)$ is a topological embedding. □

As in the classical case of the fields $\mathbb{R}$ or $\mathbb{C}$ (see [15]) we have the following property for the NA case.

Proposition 6.4. *The universal arrow $i : (X,\mathcal{U}) \to L_{\mathbb{F}}(X,\mathcal{U})$ is a closed embedding for any NA valued field $\mathbb{F}$.*

Proof. We have to show that $X = i(X)$ is closed in $L_{\mathbb{F}}(X)$. Let $v \in L_{\mathbb{F}}(X)$ be a vector such that $v \notin X$. It is enough to find a locally convex space V and a continuous linear morphism $\Phi : L_{\mathbb{F}}(X) \to V$ such that $\Phi(v) \notin cl(\Phi(X))$. For $v = \lambda x$ with $\lambda \neq 1$ and $x \in X$ consider the continuous functional

$$\Phi : L_{\mathbb{F}}(X) \to \mathbb{F}, \ \sum_{k=1}^{m} \lambda_k x_k \mapsto \sum_{k=1}^{m} \lambda_k.$$

Then $\Phi(v) = \lambda \notin cl(\Phi(X)) = \{1\}$. The same Φ works for the case of $v = \mathbf{0}$.

Now we may suppose that $v = \sum_{i=1}^{n} \lambda_i x_i$ with non-zero coefficients λ_i and that $\mathrm{supp}(v)$ contains at least two elements from X. That is, $\mathrm{supp}(u) = \{x_1, x_2, x_3, \ldots, x_n\}$, where $x_1, x_2 \in X$ and $n \geq 2$. Define V as the 2-dimensional NA normed $\mathbb{F}$-space $\mathbb{F}^2$ (with the

max ultra-norm). Since the uniform space $(X,\mathcal{U})$ is NA and Hausdorff, one may partition it into three clopen disjoint subsets

$$X = X_1 \cup X_2 \cup X_3$$

such that

$$x_1 \in X_1, x_2 \in X_2, x_k \in X_3 \ \ \forall\, 3 \leq k \leq n.$$

Now define

$$\varphi : X \to V = \mathbb{F}^2, \qquad \varphi(x) = \begin{cases} (1,0) & \text{for } x \in X_1 \\ (0,1) & \text{for } x \in X_2 \\ (0,0) & \text{for } x \in X_3. \end{cases}$$

This map is uniformly continuous and $\mathbb{F}^2$ is a locally convex NA $\mathbb{F}$-space. Hence, by the universality property, there exists the continuous extension $\Phi : L_{\mathbb{F}}(X) \to V$. Now observe that

$$\Phi(v) = (\lambda_1, \lambda_2) \notin cl(\Phi(X)) = \{(1,0),(0,1),(0,0)\}.$$

$\square$

6.1. **Normability and metrizability.**

Theorem 6.5. *Let $\mathbb{F}$ be an NA valued field with a trivial valuation, (X,d) be an ultra-metric space and $\mathcal{U}(d)$ be the uniformity of d. Then the free NA locally convex space $L_{\mathbb{F}}(X,\mathcal{U}(d))$ is normable by the Kantorovich ultra-norm $||\cdot||_d$.*

Proof. As in Lemma 4.1 consider the extension of d on $\overline{X}$. Next, by Theorem 4.3, we have the corresponding Kantorovich ultra-norm $||\cdot||$. It suffices to show that if $\varphi : (X,d) \to V$ is a uniformly continuous map to a locally convex space V, then the linear extension $\Phi : (L_{\mathbb{F}}(X), ||\cdot||) \to V$ is continuous. Being a locally convex space the topology of V is defined by a collection of ultra-seminorms $\{\rho_i\}_{i\in I}$. Clearly, $\varphi : (\overline{X},d) \to V$ is uniformly continuous. Fix $\varepsilon > 0$ and $i_0 \in I$. It follows that there exists $\delta > 0$ such that $\rho_{i_0}(\varphi(x) - \varphi(y)) < \varepsilon \ \forall x,y \in \overline{X}$ with $d(x,y) < \delta$. Now assume that $u \in L_{\mathbb{F}}(X)$ with $||u|| < \delta$. We prove the continuity of Φ by showing that $\rho_{i_0}(\Phi(u)) < \varepsilon$. By the definition of the ultra-norm $||\cdot||$ there exists a decomposition $u = \sum\limits_{i=1}^{n} \lambda_i(x_i - y_i)$ such that $\max\limits_{1\leq i\leq n} |\lambda_i| d(x_i,y_i) < \delta$. Since the valuation $|\cdot|$ is trivial we obtain $\max\limits_{1\leq i\leq n} d(x_i,y_i) < \delta$. It follows that

$$\rho_{i_0}(\Phi(u)) = \rho_i(\Phi(\sum_{i=1}^{n} \lambda_i(x_i - y_i))) = \rho_i(\sum_{i=1}^{n} \lambda_i(\varphi(x_i) - \varphi(y_i))) \leq$$
$$\leq \max_{1\leq i\leq n} |\lambda_i| \rho_{i_0}(\varphi(x_i) - \varphi(y_i)) = \max_{1\leq i\leq n} \rho_{i_0}(\varphi(x_i) - \varphi(y_i)) < \varepsilon.$$

$\square$

It is known that if a Tychonoff space X is non-discrete, then $A(X)$ is not metrizable (see [2, Theorem 7.1.20]). This result inspired us to obtain the following.

Proposition 6.6. *Let $(X,\mathcal{U})$ be a non-discrete NA uniform space. Let $\mathbb{F}$ be a complete NA valued field with a non-trivial valuation. Then $L_{\mathbb{F}}(X,\mathcal{U})$ is not metrizable.*

Proof. Assuming the contrary, there exists a decreasing sequence $\{U_n\}_{n\in\mathbb{N}}$ which forms a local base at $\mathbf{0} \in L_{\mathbb{F}}(X,\mathcal{U})$. Since the valuation $|\cdot|$ is non-trivial, there exists $\lambda \in \mathbb{F}$ with $|\lambda| > 1$. In view of Theorem 6.2 $(X,\mathcal{U})$ is a uniform subspace of $L_{\mathbb{F}}(X,\mathcal{U})$. By the continuity of the scalar multiplication it follows that there exists a sequence of entourages $\varepsilon_n \in \mathcal{U}$ such that $\lambda^n(x-y) \in U_n \ \forall x,y \in \varepsilon_n$. Since $\mathcal{U}$ is non-discrete and Hausdorff we can find a sequence $(x_n,y_n) \in \varepsilon_n$ such that $x_n \neq y_n \ \forall n \in \mathbb{N}$ and $\forall i < n \ \ x_n \notin \{x_i,y_i\}$.

Clearly, the sequence $u_n = \lambda^n(x_n - y_n) \in U_n$ converges to $\mathbf{0}$. Let us show that this leads to a contradiction. Since $(X,\mathcal{U})$ is NA it is easy to define, by induction on n, a sequence $\{f_n : n \in \mathbb{N}\}$ of uniformly continuous functions on $(X,\mathcal{U})$ with values in $\mathbb{F}$ such that for every $n \geq 1$:

(1) $|f_n(x)| \leq |\lambda|^{-n}$ $\forall x \in X$;

(2) $f_n(x_k) = f_n(y_k) = f_n(y_n) = 0_{\mathbb{F}}$ $\forall k < n$;

(3) $f_n(x_n) = \lambda^{-n} - \sum\limits_{k=1}^{n-1}(f_k(x_n) - f_k(y_n))$ if $|\sum\limits_{k=1}^{n-1}(f_k(x_n) - f_k(y_n))| \leq |\lambda|^{-n}$ and $f_n(x_n) = \lambda^{-n}$ otherwise.

By (3) and the strong triangle inequality we have $|f_n(x_n) + \sum\limits_{k=1}^{n-1}(f_k(x_n) - f_k(y_n))| \geq |\lambda|^{-n}$. By (1) for every $x \in X$ the sequence of partial sums $\left\{\sum\limits_{k=1}^{n} f_k(x)\right\}_{n\in\mathbb{N}}$ is Cauchy. Since the field $\mathbb{F}$ is complete, the function $f = \sum\limits_{n=1}^{\infty} f_n$ is well defined. From (1) it follows that f is uniformly continuous, and thus it admits, an extension to a linear continuous map $\widetilde{f} : L_{\mathbb{F}}(X,\mathcal{U}) \to \mathbb{F}$. For every $n \in \mathbb{N}$ we have

$$|\widetilde{f}(u_n)| = |\lambda|^n \cdot |\sum_{k=1}^{\infty}(f_k(x_n) - f_k(y_n))| =$$

$$= |\lambda|^n \cdot |f_n(x_n) + \sum_{k=1}^{n-1}(f_k(x_n) - f_k(y_n))| \geq |\lambda|^n \cdot |\lambda|^{-n} = 1.$$

It follows that the sequence $\{\widetilde{f}(u_n)\}$ does not converge to $\mathbf{0}$, contradicting the continuity of $\widetilde{f}$. □

In contrast, note that the uniform free NA abelian topological group $A_{\mathcal{NA}}$ (Definition 6.7) is metrizable for every metrizable NA uniform space $(X,\mathcal{U})$ (see [10] and also Remark 6.9).

6.2. **Free abelian NA groups and NA Tkachenko-Uspenskij theorem.** Recall the following definition from [10].

Definition 6.7. Let $(X,\mathcal{U})$ be an NA uniform space. The *uniform free NA abelian topological group of* $(X,\mathcal{U})$ is denoted by $A_{\mathcal{NA}}$ and defined as follows: $A_{\mathcal{NA}}$ is an NA abelian topological group for which there exists a universal uniform map $i : X \to A_{\mathcal{NA}}$ satisfying the following universal property. For every uniformly continuous map $\varphi : (X,\mathcal{U}) \to G$ into an abelian NA topological group G there exists a unique continuous homomorphism $\Phi : A_{\mathcal{NA}} \to G$ for which the following diagram commutes:

$$\begin{tikzcd}
(X,\mathcal{U}) \arrow[r, "i"] \arrow[dr, "\varphi"'] & A_{\mathcal{NA}} \arrow[d, "\Phi"] \\
 & G
\end{tikzcd}$$

Let $(X,\mathcal{U})$ be an NA uniform space and $Eq(\mathcal{U})$ be the set of all equivalence relations from $\mathcal{U}$.

Theorem 6.8. [10, Theorem 4.14] *Let $(X,\mathcal{U})$ be NA and let $\mathcal{B} \subseteq Eq(\mathcal{U})$ be a base of $\mathcal{U}$. For every $\varepsilon \in \mathcal{B}$ denote by $< \varepsilon >$ the subgroup of $A(X)$ algebraically generated by the set $\{x - y \in A(X) : (x, y) \in \varepsilon\}$. Then $\{< \varepsilon >\}_{\varepsilon\in\mathcal{B}}$ is a local base at the zero element of $A_{\mathcal{NA}}(X,\mathcal{U})$.*

Remark 6.9. It is easy to see from the description above that if (X, d) is an ultra-metric space, then $A_{\mathcal{NA}}$ is metrizable. The following theorem provides a specific metrization which can be viewed as a Graev type ultra-norm.

Lemma 6.10. *Let (X, d) be an ultra-metric space treated as an ultra-metric subspace of $(\overline{X}, d)$ as in Lemma 4.1. Then $A_{\mathcal{NA}}$ is metrizable by the Graev type ultra-norm defined as follows. For $u \in A(X)$ let*

$$||u|| := \inf \left\{ \max_{1 \leq i \leq n} d(x_i, y_i) : u = \sum_{i=1}^{n} (x_i - y_i),\ x_i, y_i \in \overline{X} \right\}.$$

Proof. Observe that for $\varepsilon < 1$ we have $B_d(\mathbf{0}, \varepsilon) =< \varepsilon >$, where $B_d(\mathbf{0}, \varepsilon)$ is the open ε-ball. □

Remark 6.11. Suppose that $(X,\mathcal{U})$ is an NA uniform space generated by a collection of ultra-seminorms $\{d_i\}_{i \in I}$. Then using the idea of Lemma 6.10 one can show that the topology of $A_{\mathcal{NA}}$ is generated by the set of the corresponding Graev type ultra-norms $\{|| \cdot ||_{d_i}\}_{i \in I}$. So we have an analogy with Theorem 6.2. At the same time we have one key difference. In the description of $A_{\mathcal{NA}}$ it is enough to consider any set of ultra-pseudometrics $\{d_i\}_{i \in I}$ which generate the uniformity $\mathcal{U}$ on X.

By Tkachenko-Uspenskij theorem [21, 22], the free abelian topological group $A(X)$ is a topological subgroup of $L(X)$ (here $\mathbb{F} = \mathbb{R}$). This can be derived (as in [22]) using the usual integer value property and descriptions of Graev's extension. Consider an NA valued field $\mathbb{F}$ of characteristic zero. It is clear that, algebraically, $A_{\mathcal{NA}}(X)$ is a natural subgroup of $L_{\mathbb{F}}(X)$ since $\mathbb{Q}$ is embedded in $\mathbb{F}$ as a subfield. So, it is natural to ask for which NA valued fields $\mathbb{F}$ we have an analogue of Tkachenko-Uspenskij theorem. Theorem 6.13 shows that this is true if and only if the valuation of $\mathbb{F}$ is trivial on $\mathbb{Q}$. First we give a particular example.

Example 6.12. Tkachenko-Uspenskij theorem is not true for the field $\mathbb{F} = \mathbb{Q}_p$ of p-adic numbers (with its standard valuation). Clearly, $\lim p^n = 0_{\mathbb{F}}$ in $\mathbb{F}$. Now, let $x, y \in X$ be a pair of distinct points in an ultra-metric space X. By the continuity of the operations $u_n := p^n(x - y)$ converges to zero in the free locally convex space $L_{\mathbb{F}}(X)$. At the same time it is not true in the free NA abelian group $A_{\mathcal{NA}}(X)$, as it follows from the internal description of the topology of $A_{\mathcal{NA}}(X)$ (see Theorem 6.8 or [10]).

Theorem 6.13. *Let $\mathbb{F}$ be an NA valued field and $(X,\mathcal{U})$ be an NA uniform space. Suppose also that $\mathrm{char}(\mathbb{F}) = 0$ and consider $A_{\mathcal{NA}}(X)$ as an algebraic subgroup of $L_{\mathbb{F}}(X)$. The following conditions are equivalent:*

(1) $A_{\mathcal{NA}}$ is a topological subgroup of $L_{\mathbb{F}}(X,\mathcal{U})$.
(2) The valuation of $\mathbb{F}$ is trivial on $\mathbb{Q}$.

Proof. (1) ⇒ (2): If the valuation on $\mathbb{Q}$ is not trivial, then by Ostrowski's Theorem 3.1 this restricted valuation is equivalent to the p-adic valuation. Now the proof is reduced to the concrete case of Example 6.12.

(2) ⇒ (1): By Proposition 6.3 we know that $L_{\mathbb{Q}}(X,\mathcal{U})$ is a topological subgroup of $L_{\mathbb{F}}(X,\mathcal{U})$. So it suffices to show that $A_{\mathcal{NA}}$ is a topological subgroup of $L_{\mathbb{Q}}(X,\mathcal{U})$. Let $\{d_i\}_{i \in I}$ be a family of ultra-pseudometrics generating the uniformity $\mathcal{U}$. For every i extend d_i to $\overline{X}$ as in Lemma 4.1. Then consider the Kantorovich ultra-seminorm (Theorem 4.3) $||\cdot||_{d_i}$ on $L_{\mathbb{Q}}(X)$. Since the restricted valuation $|\cdot|$ on $\mathbb{Q}$ is trivial, the topology of $L_{\mathbb{Q}}(X,\mathcal{U})$ is generated by the family $\{|| \cdot ||_{d_i}\}_{i \in I}$. It suffices to prove the following claim.

Claim : Let $(\overline{X}, d)$ be an ultra-pseudometric space, $||\cdot||^L$ be the corresponding Kantorovich ultra-seminorm on $L_{\mathbb{Q}}(X)$ and $||\cdot||^A$ be the corresponding Graev type ultra-seminorm on $A_{\mathcal{NA}}$ (from Lemma 6.10). Then $||u||^L = ||u||^A$ for every $u \in A(X)$.

Proof. Since $\mathbb{Z}$ is an additive subgroup of $\mathbb{Q}$, it follows by Lemma 5.1 that

$$||u||^L = \inf\left\{ \max_{1\leq i\leq n} |\lambda_i| d(x_i, y_i) : u = \sum_{i=1}^{n} \lambda_i (x_i - y_i),\ x_i, y_i \in \overline{X},\ \lambda_i \in \mathbb{Z} \right\} =$$

$$= \inf\left\{ \max_{1\leq i\leq n} d(x_i, y_i) : u = \sum_{i=1}^{n} \lambda_i (x_i - y_i),\ x_i, y_i \in \overline{X},\ \lambda_i \in \mathbb{Z} \right\} =$$

$$= \inf\left\{ \max_{1\leq i\leq n} d(x_i, y_i) : u = \sum_{i=1}^{n} (x_i - y_i),\ x_i, y_i \in \overline{X} \right\} = ||u||^A.$$

□

□

Example 6.14. Theorem 6.13 can be applied to the Levi-Civita field $\mathcal{R}$. Indeed, as it was noted in Example 3.2, $\mathcal{R}$ admits a natural dense valuation. Its restriction on $\mathbb{Q}$ is trivial. We conclude, by Theorem 6.13, that $A_{\mathcal{NA}}$ is a topological subgroup of $L_{\mathcal{R}}(X, \mathcal{U})$ for every NA uniform space $(X, \mathcal{U})$.

7. Pointed version and the dual space

Using similar techniques to those mentioned in the previous sections, one can study the pointed version of NATP. However, its connection to the dual space is a unique feature which we present below.

Let (X, d, e) be a pointed ultra-pseudometric with a base point e. Let $L_{\mathbb{F}}(X)$ be the free pointed $\mathbb{F}$-vector space on the pointed set (X, e). As before let

$$L^0_{\mathbb{F}}(X) := \left\{ \sum_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X) |\ \sum_{i=1}^{n} \lambda_i = 0_{\mathbb{F}} \right\}.$$

Definition 7.1. The *Kantorovich ultra-seminorm* is the ultra-seminorm on $L^0_{\mathbb{F}}(X)$ given by the following formula. For $u \in L^0_{\mathbb{F}}(X)$ let

$$||u|| := \inf\left\{ \max_{1\leq i\leq n} |\lambda_i| d(x_i, y_i) : u = \sum_{i=1}^{n} \lambda_i (x_i - y_i),\ x_i, y_i \in X,\ \lambda_i \in \mathbb{F} \right\}.$$

It follows from the definition of the Kantorovich ultra-seminorm that $||x - y|| \leq d(x, y)$ for every $x, y \in X$. As in the non-pointed case we can show that $||x-y|| = d(x, y)$ and $||\cdot||$ is an ultra-norm whenever d is an ultra-metric. It is well known that the map $x \mapsto x - e$ defines an isometric embedding of a metric space (X, d) into the classical Arens-Eells space. See, for example, [25, Section 2.2]. One may show that the same rule defines an isometric embedding of a pointed ultra-metric space (X, d, e) into $(L^0_{\mathbb{F}}(X), ||\cdot||)$. For every pointed Lipschitz function $f : X \to \mathbb{F}$ we have a canonically defined continuous functional $L^0_{\mathbb{F}}(X) \to \mathbb{F}$. Moreover, for a nontrivially valued NA field $\mathbb{F}$, the dual NA Banach space of $L^0_{\mathbb{F}}(X)$ can be identified with the NA Banach space Lip_0 of all pointed Lipschitz functions $f : X \to \mathbb{F}$. We omit the verification which essentially is very similar to the arguments of [25, Theorem 2.2.2]. Note that the nontriviality of the valuation is important in order to ensure that every continuous functional $L^0_{\mathbb{F}}(X) \to \mathbb{F}$ is a Lipschitz function. See [18, Prop. 3.1].

8. Appendix

Let (X, d) be a pseudometric space and $\mathbb{C}$ be the field of complex numbers. As in the case of reals (Equation (2.2)) define the Kantorovich seminorm on $L^0_{\mathbb{C}}(X)$ as follows. For every $v \in L^0_{\mathbb{C}}(X)$

$$||v|| = \inf\left\{\sum_{i=1}^{l} |\rho_i| d(a_i, b_i) : v = \sum_{i=1}^{l} \rho_i (a_i - b_i),\ \rho_i \in \mathbb{C}, a_i, b_i \in X\right\}. \tag{8.1}$$

The following was mentioned in [5, 25, 6].

Theorem 8.1. *Support elements do not determine the Kantorovich norm for the field $\mathbb{F} := \mathbb{C}$ of complex numbers.*

The following example (which appears in [5, p. 90], [25, Ex. 1.5.7, p. 18] without details) implies Theorem 8.1. That is, in general, the infimum in (8.1) cannot be achieved or even approximated by support elements. This was mentioned also in [6]. As we show below, one may say even more: in this example the infimum is attained outside the support.

Example 8.2. Let $X = \{e, p, q, r\}$ and d be a metric on X defined as follows: $d(p, q) = d(p, r) = d(q, r) = 1$ and $d(e, p) = d(e, q) = d(e, r) = \frac{1}{2}$. Let $\lambda = 1 \cdot p + \mu q + \nu r \in L^0_{\mathbb{C}}(X)$, where $S = \{1, \mu, \nu\}$ denotes the set of the three complex cube roots of unity. We show that the infimum in the definition of $||\lambda||$ cannot be achieved or even approximated by support elements. We also show that the infimum is attained outside the support and that $||\lambda|| = \frac{3}{2}$.

(a) Since $1 + \mu + \nu = 0$ we have $\lambda = (p - e) + \mu(q - e) + \nu(r - e)$. It follows that $||\lambda|| \le d(p, e) + |\mu| d(q, e) + |\nu| d(r, e) = \frac{3}{2}$. We will show that the minimal sum-cost which comes from presentations of λ that include only support elements, is strictly larger than $\frac{3}{2}$. When dealing with support elements it suffices to consider presentations of λ of the form $\lambda = c_{pq}(p - q) + c_{pr}(p - r) + c_{qr}(q - r)$ where $c_{pq}, c_{pr}, c_{qr} \in \mathbb{C}$. Indeed, this follows from the reduction rules:
(1) Replace $m(x - y)$ with $-m(y - x)$.
(2) Replace the terms $m(x - y),\ n(x - y)$ with $(m + n)(x - y)$.
If $\lambda = c_{pq}(p - q) + c_{pr}(p - r) + c_{qr}(q - r)$ then $c_{pq} + c_{pr} = 1$, $-c_{pq} + c_{qr} = \mu$, $-c_{pr} - c_{qr} = \nu$. So, the infimum is

$$\inf\{|c_{pq}| d(p, q) + |c_{pr}| d(p, r) + |c_{qr}| d(p, r) : c_{pq} + c_{pr} = 1, -c_{pq} + c_{qr} = \mu, -c_{pr} - c_{qr} = \nu\}.$$

Taking into account that $d(p, q) = d(p, r) = d(q, r) = 1$, we solve the system of linear equations and see that the latter expression is equal to $\inf_{t \in \mathbb{C}}(|\mu - t| + |0 - t| + |-\nu - t|)$. Finding this infimum is a simple geometrical problem since $0, \mu, -\nu$ are three vertices of an equilateral triangle in the complex plane. It follows that the infimum is equal to $\sqrt{3}$. Clearly $\sqrt{3} > \frac{3}{2}$ as needed.

(b) We will show that the infimum is attained outside the support and that $||\lambda|| = \frac{3}{2}$. We already know that there exists a presentation of λ for which the value of the sum-cost is $\frac{3}{2}$. So $||\lambda|| \le \frac{3}{2}$ and it suffices to show that $||\lambda|| \ge \frac{3}{2}$. This is done by showing that for every presentation of λ of the form $\lambda = c_{ep}(e - p) + c_{eq}(e - q) + c_{er}(e - r) + c_{pq}(p - q) + c_{pr}(p - r) + c_{qr}(q - r)$, where $c_{ep}, c_{eq}, c_{er}, c_{pq}, c_{pr}, c_{qr} \in \mathbb{C}$, we have

$$|c_{ep}| d(e, p) + |c_{eq}| d(e, q) + |c_{er}| d(e, r) + |c_{pq}| d(p, q) + |c_{pr}| d(p, r) + |c_{qr}| d(q, r) =$$

$$= \frac{1}{2}(|c_{ep}| + |c_{eq}| + |c_{er}|) + |c_{pq}| + |c_{pr}| + |c_{qr}| \ge \frac{3}{2}.$$

We compare the coefficients of e, p, q, r in the normal presentation of λ and in the "new" presentation and obtain
(1) $c_{ep} + c_{eq} + c_{er} = 0$,
(2) $-c_{ep} + c_{pq} + c_{pr} = 1$,
(3) $-c_{eq} - c_{pq} + c_{qr} = \mu$,
(4) $-c_{er} - c_{pr} - c_{qr} = \nu$.

Now, using the triangle inequality and properties (2) – (4), we obtain

$$\frac{1}{2}(|c_{ep}|+|c_{eq}|+|c_{er}|)+|c_{pq}|+|c_{pr}|+|c_{qr}| = \frac{1}{2}(|-c_{ep}|+|c_{pq}|+|c_{pr}|)+\frac{1}{2}(|-c_{eq}|+|-c_{pq}|+|c_{qr}|)+$$
$$+\frac{1}{2}(|-c_{er}|+|-c_{pr}|+|-c_{qr}|) \geq \frac{1}{2}(|-c_{ep}+c_{pq}+c_{pr}|+|-c_{eq}-c_{pq}+c_{qr}|+|-c_{er}-c_{pr}-c_{qr}|) =$$
$$= \frac{1}{2}(|1| + |\mu| + |\nu|) = \frac{3}{2}.$$

We showed that in the archimedean case the infimum can be attained outside the support. In fact, as the following example shows, sometimes the infimum is not attained at all.

Example 8.3. Let $X = \{p, q, r\}$ and d be a metric on X such that $d(p, q) = d(p, r) = d(q, r) = 1$. Let $\mathbb{F} = \mathbb{Q}(i)$ be the subfield of $\mathbb{C}$, where $\mathbb{Q}(i) := \{a + bi : \ a, b \in \mathbb{Q}\}$. We will show that the infimum in the definition of $||u||$ is not attained in $\mathbb{F}$ for $u = (1-i)p+iq-r$. It suffices to show that the infimum

$$\inf\{|c_{pq}|d(p,q)+|c_{pr}|d(p,r)+|c_{qr}|d(q,r) : c_{pq}+c_{pr} = 1-i, \ -c_{pq}+c_{qr} = i, \ -c_{pr}-c_{qr} = -1\} =$$
$$= \inf_{t\in\mathbb{F}}(|t| + |t-i| + |t-1|)$$

is not attained. Since $\mathbb{F} = \mathbb{Q}(i)$ is a dense subfield of $\mathbb{C}$ it follows that the latter expression is equal to $\inf_{t\in\mathbb{C}}(|t-i| + |1-t| + |t|)$. This infimum is attained at a unique point $p \in \mathbb{C}$ that is the Fermat-Torricelli point of the triangle in the complex plane with vertices $0, 1, i$. One can show that $p \notin \mathbb{Q}(i)$. By the uniqueness of p it follows that the infimum in the definition of $||u||$ is not attained in $\mathbb{F}$.

9. Some possible developments and problems

(1) One of the most attractive directions is the study of concrete applications of NATP (non-archimedean transportation problem).
(2) A natural perspective is to extend the discrete version of NATP to a *continuous* one (which in the classical case is based on measures).
(3) It would be interesting to look for additional properties of the free NA locally convex $\mathbb{F}$-space.

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